CARTESIAN APPROACH FOR THE HEAVY RIGID BODY IN THE SUSLOV AND VESELOV CASE

Rafael Ramírez[1] and Natalia Sadovskaia[2]

[2] Departamento de Ingenería Informática
Universitat Rovira i Virgili de Tarragona
rafaelorlando.ramirez@ urv.cat
[2] Departamento de Matemática Aplicada II
Universitat Politècnica de Catalunya
natalia.sadovskaia@upc.edu

Abstract

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In the history of mechanics, there have been two points of view for studying mechanical systems: The Newtonian and the Cartesian. The Cartesian point of view affirms (by using the modern mathematical language) that it is possible to solve the dynamics problem inside the configuration space. In this paper we develop the Cartesian approach for mechanical systems with constraints which are linear with respect to velocity. The obtained results are illustrated into the study of the three problem: the behavior of the heavy rigid body in the Suslov and Veselov case and the rattleback . The first problem concerns the inertial rotation of a rigid body about a fixed point with a non-holonomic constraints, i.e., the projection of the angular velocity on a certain straight line fixed to the body is equal to zero. The Veselov problem is analogous to the Suslov problem but in this case the projection of the angular velocity is in the fixed exes in the space. The thirst problem consist into the study a convex asymmetric rigid body rolling without sliding on a horizontal plane (rattleback).

Key words: Non-holonomic systems, Cartesian approach, Newtonian approach, constraint, differential equation, Lagrangian systems, Suslov's problem, Veselov's problem, rattleback, .

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1. Introduction

In the history of mechanics, there have been two points of view for studying mechanical systems: The Newtonian and the Cartesian. In "Philosophiae Naturalis Principia Mathematica" (1687), Newton considers that movements of celestial bodies can be described by differential equations of the second order. To determine their trajectory, it is necessary to give the initial position and velocity.

Descartes proposed that the behavior of the celestial bodies be studied from another point of view. These ideas were stated in "Principia Philosophiae" (1644) and in "Discours de la métode" (1637). According to Descarte the understanding of cosmology starts from acceptance of the initial chaos, whose moving elements are ordered according to certain fixed laws and form the Cosmo. He consider that the Universe is filled with a tenuous fluid matter (ether), which is constantly in a vortex motion. This motion moves the largest particle of matter of the vortex axis, and they subsequently form planets. Then, according to what Descartes wrote in his "Treatise on Light", "the material of the Heaven must be rotate the planets not only about the Sun but also about their own centers...and this will hence form several small Heavens rotating in the same direction as the great Heaven."

Newton gave a simpler, but stronger, argument against Descarte's theory. If the Descarte's ideas is correct, bodies are carried by the ether, and the equations of motion are consequently of first order: the velocity of a particle depend only on its position. However, Newton noted that some of the observed comets move in a direction opposite to that of all the planets [Kozlov1].

In the modern scientific literature the study of the Descarte ideas we can find in the monographic of V.V. Kozlov in which the author said "In the present book, one more attempt is made to rehabilitate Descarte's vortex theory..." . In this books, Kozlov affirms "solving dynamics problem is possible inside the configuration space".

As we observe , the equation of motion in the Descartes theory must be of the first order

$$\dot{\mathbf{x}} = \mathbf{v}(x) \tag{1.1}$$

Hence, to determine the trajectory from Descartes's point of view it is necessary to give only the initial position. Descarte gave no principles for constructing the field $\mathbf{v}$ for different mechanical systems.

A main achievement of Newton was perceiving that the dynamics of real systems are described by second-order differential equations. To deduce the equations of motion to the investigation of a dynamics systems (i.e., to first order equation), it is necessary to double the dimension of the position space and to introduce the auxiliary phase space. However, we are interested not in the phase trajectories themselves but in their projection on the configuration space.

Definition

The vector field (1.1) we shall call the Cartesian vector field.

The aim of the present paper is to develop the Descarte ideas for mechanical systems with constraints which are linear with respect to the velocity.

2. CONSTRUCTION THE CARTESIAN VECTOR FIELD FOR NON-HOLONOMIC SYSTEM

Firstly we shall introduce the following notation and concept.

Let $\mathcal{Q}$ be a smooth manifold of the dimension N with local coordinates $x = (x^1, ..., x^N)$ and equipped by the Riemann metric $G = (G_{kj}(x))$.

By $\xi(\mathcal{Q})$, $\Lambda(\mathcal{Q})$, ∇ we denote respectively the Lie algebra of vector fields on $\mathcal{Q}$ and the algebra of the 1-form on $\mathcal{Q}$, and the connection:

$$\nabla : \xi(\mathcal{Q}) \times \xi(\mathcal{Q}) \longmapsto \xi(\mathcal{Q})$$
$$(u, v) \longmapsto \nabla_u v$$

which is R lineal with respect to v and C^∞ lineal with respect to u and is compatible with metric G, i.e., $\nabla_u G(v, w) = 0, \quad \forall u, v, w \in \xi(\mathcal{Q})$.

Let $\mathbf{v} \in \xi(\mathcal{Q})$ be a vector field:

$$\mathbf{v} = det \begin{pmatrix} \Omega_1(\partial_1) & \Omega_1(\partial_2) & \dots & \Omega_1(\partial_N) & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_M(\partial_1) & \Omega_M(\partial_2) & \dots & \Omega_M(\partial_N) & 0 \\ \Omega_{M+1}(\partial_1) & \Omega_{M+1}(\partial_2) & \dots & \Omega_{M+1}(\partial_N) & \lambda_{M+1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_N(\partial_1) & \Omega_N(\partial_2) & \dots & \Omega_N(\partial_N) & \lambda_N \\ \partial_1 & \partial_2 & \dots & \partial_N & 0 \end{pmatrix},$$

where $\partial_k = \frac{\partial}{\partial x^k}$, we shall consider that $\Omega_1, \Omega_2, ...\Omega_M, \quad M \leq N-1$ are given 1-forms, and $\Omega_{M+1}, \Omega_{M+2}, ...\Omega_N$, are arbitrary 1-forms on $\mathcal{Q}$. Furthermore, we assume that they are pointwise independent i.e.

$$\Upsilon \equiv \Omega_1 \wedge \Omega_2 ... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N) \neq 0,$$

The functions $\lambda_j, \quad j = M+1, ..., N$ are arbitrary functions on $\mathcal{Q}$

The vector field $\mathbf{v}$ has the following properties

1.

$$\begin{cases} \Omega_j(\mathbf{v}) = 0, & j = 1, 2, .., M \\ \Omega_j(\mathbf{v}) = -\Upsilon\lambda_j, & j = M+1, .., N. \end{cases}$$

2.The vector $\mathbf{v}(x) = (v^1(x), ..., v^N(x))^T$ can be represented as follows

$$\mathbf{v}(x) = \mathcal{M}^{-1}\lambda,$$

where $\mathcal{M} = \Big(\Omega_j(\partial_k)_{j,k=1,..N}\Big), \quad \lambda = -\Upsilon\,(0, .., 0, \lambda_{M+1}, ..\lambda_N)$. or, what is the same,

$$\mathbf{v} = \sum_{j=M+1}^{N} \lambda_j X_j \tag{2.1}$$

where X_j, $j = M+1, ..., N$ constitute a maximal set of independent vector fields on $\mathcal{Q}$ satisfying the constraints, in the sense that the components of X_j satisfy the equations

$$\Omega_j(X_k) = 0, \quad j = 1, .., M, \; k = M+1, .., N$$

3. Let σ be the 1-form associated with the vector field $\mathbf{v}$, i.e.,

$$\sigma = (\mathbf{v}(x), dx) \equiv \sum_{j,k=1}^{N} G_{jk}(x)v^j(x)dx^k \equiv \sum_{k=1}^{N} p_k dx^k$$

then the 2-form $d\sigma$:

$$d\sigma = \frac{1}{2}\sum_{j,k=1}^{N} a_{jk}(x)\Omega_j \wedge \Omega_k,$$

where $A = (a_{jk})$ is a matrix such that

$$a_{jk} = (-1)^{j+k-1}\frac{1}{\Upsilon}d\sigma \wedge \Omega_1 \wedge ... \wedge \widehat{\Omega}_k .... \wedge \widehat{\Omega}_j .... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N)$$

$\widehat{\Omega}_j$, $\widehat{\Omega}_k$ means that these elements are omitted.

It is clear that the contraction of $d\sigma$ along $\mathbf{v}$ is

$$\iota_{\mathbf{v}} d\sigma = \sum_{j=1}^{N} \Lambda_j \Omega_j,$$

where

$$\Lambda \equiv col(\Lambda_1, \Lambda_2, ..., \Lambda_N) = A^T\lambda.$$

We shall analyze the differential equations

$$\dot{\mathbf{x}} = \mathcal{M}^{-1}\lambda = \sum_{j=M+1}^{N} \lambda_j X_j \tag{2.2}$$

under the conditions

$$\begin{cases} \Lambda_j = 0, \quad j = M+1, .., N \\ \Upsilon = \Omega_1 \wedge \Omega_2 .... \wedge \Omega_N(\partial_1,\, \partial_2,\, ..., \partial_N) \neq 0 \end{cases} \tag{2.3}$$

In particular, for a constrained particle in R^3 we have that the first condition in (2.3) holds if

$$\Omega_1(rot\mathbf{v}) = 0.$$

Corollary 2.1

For the case when $M = N - 1$ the vector field $\mathbf{v}$ takes the form

$$\mathbf{v} = \lambda_N det \begin{pmatrix} \Omega_1(\partial_1) & \Omega_1(\partial_2) & \dots & \Omega_1(\partial_N) \\ \vdots & \vdots & \dots & \vdots \\ \Omega_M(\partial_1) & \Omega_M(\partial_2) & \dots & \Omega_M(\partial_N) \\ \Omega_{M+1}(\partial_1) & \Omega_{M+1}(\partial_2) & \dots & \Omega_{M+1}(\partial_N) \\ \vdots & \vdots & \dots & \vdots \\ \Omega_{N-1}(\partial_1) & \Omega_{N-1}(\partial_2) & \dots & \Omega_{N-1}(\partial_{N-1}) \\ \partial_1 & \partial_2 & \dots & \partial_N \end{pmatrix},$$

where λ_N is an arbitrary function.

The conditions (2.3) hold if and only if

$$\Upsilon = \Omega_1 \wedge \Omega_2 .... \wedge \Omega_N(\partial_1,\, \partial_2,\, ..., \partial_N) \neq 0$$

where Ω_N is an arbitrary 1-form.

Proposition 2.1 *The differential equations*

$$\dot{\mathbf{x}} = \mathbf{v}(x), \quad x \in X$$

are invariant relationship of the Lagrangian equations with Lagrangian function

$$L_0 = \frac{1}{2}||\dot{\mathbf{x}} - \mathbf{v}(\mathbf{x})||^2 \equiv \frac{1}{2}\sum_{j,k=1}^{N} G_{kj}(x)(\dot{x}^j - v^j(x))(\dot{x}^k - v^k(x))$$

In fact, by derivation we deduce that $\nabla_{\dot{\mathbf{x}}}(\dot{\mathbf{x}} - \mathbf{v}(x)) = 0$, or,

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{\mathbf{x}}} L_0) = 0,$$

which are equivalent to Lagrangian equations with the Lagrangian function L_0 given above. It is easy to show that these equations admits the representation

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{x}^j} T) = \omega(\partial_j) + \nabla_{\dot{\mathbf{x}} - \mathbf{v}(\mathbf{x})} p_j$$

where

$$T = \frac{1}{2}||\dot{\mathbf{x}}||^2, \quad p_j = \sigma(\partial_j)$$
$$\omega = d\frac{||\mathbf{v}||^2}{2} + \iota_{\mathbf{v}} d\sigma,$$

σ is the 1- form associated with the vector field $\mathbf{v}$.

We shall study the case when (2.2) and (2.3) hold. The differential equations which describe the behavior of such mechanical systems under these restrictions can be represented as follows

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{x}^k} T) = \frac{\partial \frac{1}{2}||\mathbf{v}||^2}{\partial x^k} + \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_k), \quad k = 1, 2, ..N, \tag{2.4}$$

and can be interpreted as the equations of motion of non-holonomic mechanical systems with an active potential field of force with potential U :

$$U = \frac{1}{2}||\mathbf{v}(x)||^2 + U_0, \quad U_0 = const.$$

and with the reactive forces with the components

$$(\sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_1), \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_2), ..., \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_N)),$$

generated by the constraints

$$\Omega_j(\dot{\mathbf{x}}) \equiv \sum_{k=1}^{N} \Omega_j(\partial_k)\dot{x}^k = 0, \quad j = 1, 2, ..N.$$

Corollary 2.2
If

$$\begin{cases} M = N - 1 \\ \Omega_j = df_j(x), \quad j = 1, 2, ..., N - 1 \\ \Omega_N = df_N \end{cases}$$

Then the equations (2.2)+(2.3) and (2.4) take the form respectively

$$(2.5) \qquad \begin{cases} \dot{\mathbf{x}} = \lambda_N \det \begin{pmatrix} df_1(\partial_1) & \dots & df_1(\partial_N) \\ \vdots & & \vdots \\ df_{N-1}(\partial_1) & \dots & df_{N-1}(\partial_N) \\ \partial_1 & \dots & \partial_N \end{pmatrix} \\ \Upsilon = df_1 \wedge df_2 .... \wedge df_N(\partial_1, \partial_2, ..., \partial_N) \neq 0 \end{cases}$$

$$(2.6) \qquad \nabla_{\dot{\mathbf{x}}}(\partial_{\dot{x}^k} T) = \frac{\partial \frac{1}{2}||\mathbf{v}||^2}{\partial x^k} + \sum_{j=1}^{N-1} \lambda_N a_{Nj}(x) df_j(\partial_k).$$

where λ_N is an arbitrary function.

Definition

The studying of the behavior of the non-holonomic systems by using the equations (2.2)+(2.3) or (2.4) we called *Cartesian and Lagrangian approach* respectively [Sad, Ram] and by applying the equations deduced from the D'Alembert-Lagrange Principle we called the *Classical approach.*

With respect to the proposed us approach we have the following conjectures.

Conjecture

The Cartesian and Lagrangian approach are equivalent.

This conjecture supported the following facts. First, the solutions of (2.2)+(2.3) are solutions of (2.4) in view of proposition 2.1. Second, the solutions of the equations (2.4) depend on the $2N - M$ initial conditions. The solutions of (2.2) depend on N initial conditions and $N - M$ functions which are solutions of the linear partial differential equations of first order (2.3).

To illustrate this conjecture we study the following example.

Example 1

A NON-HOLONOMICALLY CONSTRAINED PARTICLE IN $\mathbb{R}^3$.

Consider a particle with the Lagrangian (3.4) and non-holonomic constraints

$$\Omega_1(\dot{\mathbf{x}}) \equiv \dot{x} + a(z)\dot{y} = 0$$

This instructive academic example, in the particular case when $a(z) = z$, due to Rosenberg [Ros]. This example was also used to illustrate the theory in Bates and Sniatycki [Bates]. The Cartesian approach in this case produce the following vector field $\mathbf{v}$:

$$\mathbf{v} = \lambda_2(a(z)\partial_x - \partial_y) - \lambda_3\partial_z = \lambda_2 X_2 + \lambda_3 X_3 \tag{2.7}$$

and condition (2.3) for this case takes the form

$$\lambda_2\Omega_1(rot\mathbf{v}) = \frac{1}{2}\partial_z((1+a^2)\lambda_2^2) + (a\partial_x\lambda_3 - \partial_y\lambda_3)\lambda_2 = 0. \tag{2.8}$$

The vector field X_2 and X_3 are such that

$$\begin{cases} X_2 = a(z)\partial_x - \partial_y \\ X_3 = \partial_z \\ [X_3, X_2] = \partial_z a(z)\partial_x \end{cases} \tag{2.9}$$

We shall study the case when in (2.8)

$$\lambda_2 = \frac{A}{\sqrt{a^2+1}}, \qquad \lambda_3 = b_2(z),$$

for A an arbitrary constant and b_2 an arbitrary function.

The equations generated by the vector field $\mathbf{v}$ in this case are

$$\begin{cases} \dot{x} = \dfrac{a(z)A}{\sqrt{1+a^2(z)}} \\ \dot{y} = -\dfrac{A}{\sqrt{1+a^2(z)}} \\ \dot{z} = -b_2(z) \end{cases}$$

Hence the all trajectories of these equations are the following

$$\begin{cases} x = x_0 - A\displaystyle\int_{z_0}^{z} \frac{a(z)dz}{b_2(z)\sqrt{1+a^2(z)}} \\ y = y_0 - A\displaystyle\int_{z_0}^{z} \frac{dz}{b_2(z)\sqrt{1+a^2(z)}} \\ t = t_0 - \displaystyle\int_{0}^{z} \frac{dz}{b_2(z)} \end{cases} \tag{2.10}$$

The equation (2.4) may be rewritten as

$$\begin{cases} \ddot{x} = \partial_z(\dfrac{Aa(z)}{\sqrt{a^2(z)+1}}) \\ \ddot{y} = a(z)\partial_z(\dfrac{Aa(z)}{\sqrt{a^2(z)+1}}) \\ \ddot{z} = \partial_z b_2(z) \end{cases}$$

Corollary 2.3
All the trajectories of the equation of motion of the constrained Lagrangian system

$$< \mathrm{R}^3,\, L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(z),\, \{\dot{x} + a(z)\dot{y} = 0\} >$$

can be obtained from (2.10) [Sad].

In this example the Cartesian, the lagrangian and Classical approach coincide.

With respect to the arbitrary 1-form we posed the following problem.

Problem
Determine the 1-form $\Omega_{M+1}, ...\Omega_N$ from the condition that the smallest Lie algebra of vector fields on $\mathcal{Q}$ that contains the vector field $X_{M+1}, ..., X_N$ is finite dimensional.

If we assume that

$$X_1, X_2, .., X_M, X_{M+1}, .., X_N, ..., X_S$$

is a basis of this Lie algebra then

$$[X_j,\, X_k] = \sum_{m=1}^{S} C_{jk}^m X_m, \quad j, k = 1, 2, ..S$$

where $X_j \in \xi(\mathcal{Q})$, $j = 1, 2, ..S$ and $[X, Y]$ is the Lie brackets of vector field X and Y, and C_{jk}^m are the structure constants.

When the algebra is three dimensional then from the Bianchi representation we obtain:

$$\begin{cases} [X_1, X_2] = aX_2 + b_3 X_3 \\ [X_2, X_3] = b_1 X_1 \\ [X_3, X_1] = b_2 X_2 - aX_3 \end{cases} \tag{2.11}$$

where a, b_1, b_2, b_3 are certain constants

Example 2
In the example 1 we obtain a finite Lie algebra if in (2.9) the function a is such that

$$1)\, a(z) = z, \quad 2)\, a(z) = \exp z, \quad 3)\, a(z) = \cos z,$$

A brief calculation shows that for the first case

$$\begin{cases} X_1 = \partial_z, \quad X_2 = z\partial_x - \partial_y, \quad X_3 = \partial_x \\ [X_1, X_2] = X_3, \quad [X_1, X_3] = 0, \quad [X_2, X_3] = 0 \end{cases}$$

which correspond to the Heisenberg algebra [Bloch].

For the second case we have

$$\begin{cases} X_1 = \partial_z, \quad X_2 = \exp z\partial_x - \partial_y, \quad X_3 = \partial_y \\ [X_1, X_2] = X_3 + X_1, \quad [X_1, X_3] = 0, \quad [X_2, X_3] = 0 \end{cases}$$

For the case when $a(z) = \cos z$ by introducing the vector field X_1, X_2, X_3, X_4 :

$$\begin{cases} X_1 = -\sin z\partial_x - \partial_y \\ X_2 = \cos z\partial_x - \partial_y \\ X_3 = \partial_z \\ X_4 = \partial_y \end{cases}$$

we deduce the four dimensional Lie algebra:

$$\begin{cases} [X_2, X_3] = X_3 + X_4 \\ [X_3, X_1] = -X_2 - X_4 \\ [X_2, X_1] = 0, \quad [X_2, X_4] = 0, \quad [X_3, X_4] = 0, \quad [X_1, X_4] = 0 \end{cases}$$

3. CARTESIAN APPROACH FOR NON-HOLONOMIC SYSTEM WITH THREE DEGREE OF FREEDOM AND ONE CONSTRAINTS .

The case when $dim\mathcal{Q} = 3$ and $M = 1$ is of specific interest. We consider a natural mechanical system with configuration space $\mathcal{Q}$ and kinetic energy

$$T = \frac{1}{2}\sum_{k,j=1}^{3} G_{kj}(x)\dot{x}^j\dot{x}^k$$

Obviously, in this case the 1-form $\iota_{\mathbf{v}}d\sigma$ can be represented as follow

$$\iota_{\mathbf{v}}d\sigma = \Lambda_1\Omega_1 + \Lambda_2\Omega_2 + \Lambda_3\Omega_3$$

where Λ_j, $j = 1, 2, 3$:

$$\begin{cases} \Lambda_1 = \Omega_2 \wedge \Omega_3(\mathbf{v}, rot\mathbf{v}) \\ \Lambda_2 = \lambda_3\Omega_1(rot\mathbf{v}) \\ \Lambda_3 = -\lambda_2\Omega_1(rot\mathbf{v}) \\ rot\mathbf{v} = \frac{1}{\sqrt{detG}}\big((\partial_y p_3 - \partial_z p_2)\partial_x + (\partial_z p_1 - \partial_x p_3)\partial_y + (\partial_x p_2 - \partial_y p_1)\partial_z, \big) \\ p_k = \sum_{j=1}^{3} G_{kj}v^j, \ k = 1, 2, 3 \end{cases}$$

The system (2.2)+(2.3) take the form respectively

$$\dot{\mathbf{x}} = [\Omega_1 \times, \lambda_2\Omega_3 - \lambda_3\Omega_2] \equiv \lambda_2 X_2 + \lambda_3 X_3 \tag{3.1}$$

$$\begin{cases} \Upsilon \neq 0 \\ \Omega_1(rot\mathbf{v}) = 0 \end{cases} \tag{3.2}$$

where $\Omega_j(x) = (\Omega_j(\partial_x), \Omega_j(\partial_x)\Omega_j(\partial_x))$, $j = 1, 2, 3$; $[\quad \times, \quad]$ is the vector product on R^3.

Definition

The vector field $\mathbf{v}$ we call the Kummer vector field if

$$[\mathbf{v} \times rot\mathbf{v}] = \mathbf{0}$$

It is easy to show that the equations of motion (2.4) for $N = 3$ under the condition that $\mathbf{v}$ is a Kummer a vector field can be represented in Lagrangian form.

Example 2

HEAVY RIGID BODY IN THE SUSLOV CASE

In this section we study one classical problem of non-holonomic dynamics formulated by Suslov [Koz2]. In this problem we consider the rotational motion of a rigid body around a fixed point and subject to the non-holonomic constraints $(\mathbf{a}, \omega) = 0$ where ω is a body angular velocity and $\mathbf{a}$ is a constant vector. Suppose the body rotates in an force field with potential $U(\gamma_1, \gamma_2, \gamma_3)$. Applying the method of Lagrange multipliers we write the equations of motion in the form

$$\begin{cases} I\dot{\omega} = [I\omega \times \omega] + [\gamma \times \frac{\partial U}{\partial \gamma}] + \mu\mathbf{a} \\ \dot{\gamma} = [\gamma \times \omega] \\ (\mathbf{a}, \omega) = 0 \end{cases}$$

Where

$$I = diag(I_1, I_2, I_3),$$
$$\gamma = (\gamma_1 = \sin z \sin x, \quad \gamma_2 = \sin z \cos x, \quad \gamma_3 = \cos z)$$

I_1, I_2, I_3 are the inertial moment of the body.

If we assume that the vector $\mathbf{a} = (0, 0, 1)$ [Koz2], then

$$\begin{cases} I_1\dot{\omega}_1 = \gamma_3\partial_{\gamma_2}U - \gamma_2\partial_{\gamma_3}U \\ I_2\dot{\omega}_2 = \gamma_1\partial_{\gamma_3}U - \gamma_3\partial_{\gamma_1}U \\ (I_1 - I_2)\omega_1\omega_2 + \gamma_2\partial_{\gamma_2}U - \gamma_2\partial_{\gamma_1}U + \mu = 0 \\ \dot{\gamma}_1 = -\gamma_3\omega_2 \\ \dot{\gamma}_2 = \gamma_3\omega_1 \\ \dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1 \end{cases} \tag{3.3}$$

The above system has two independent first integrals

$$K_1 = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2) - U(\gamma_1, \gamma_2, \gamma_3)$$
$$K_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$$

By the Jacobi's theorem about the last multiplier, if there exits a third independent first integral K_3 which is functionally independent together with K_1 and K_2, then the Suslov problem is integrable by quadratures [Koz2, Mac]

To determine the integrable cases of the Suslov problem seems interesting the following result which we can prove after straightforward calculations.

Proposition 3.1

Let us suppose that the potential function U in (3.3)is determine as follows

$$U = \frac{1}{2I_1I_2}(I_1\mu_1^2 + I_2\mu_2^2) - h \tag{3.4}$$

where μ_1, $\mu_2$are solutions of the partial differential equations

$$\gamma_3(\frac{\partial\mu_1}{\partial\gamma_2} - \frac{\partial\mu_2}{\partial\gamma_1}) - \gamma_2\frac{\partial\mu_1}{\partial\gamma_3} + \gamma_1\frac{\partial\mu_2}{\partial\gamma_3} = 0 \tag{3.5}$$

Then the equations (3.3)+(3.4) admits the first integrals

$$I_1\omega_1 = \mu_2, \quad I_2\omega_2 = -\mu_1 \tag{3.6}$$

The aim of this apartat is to propose the Cartesian approach for heavy rigid body in the Suslov case.

Let us suppose that $\mathcal{Q} = SO(3)$, with the Riemann metric

$$G = \begin{pmatrix} I_3 & I_3\cos z & 0 \\ I_3\cos z & (I_1\sin^2 x + I_2\cos^2 x)\sin^2 z + I_3\cos^2 z & (I_1 - I_2)\sin x\cos x\sin z \\ 0 & (I_1 - I_2)\sin x\cos x\sin z & I_1\cos^2 x + I_2\sin^2 x \end{pmatrix}$$

$detG = I_1I_2I_3\sin^2 z,$

The given 1-form is the following $\Omega_1 = dx + \cos z dy$. By choosing the 1-form Ω_2, Ω_3 as follow

$$\Omega_2 = dy, \quad \Omega_3 = dz$$

we obtain that $\Upsilon = 1$. Hence the vector field $\mathbf{v}$ is such that

$$\mathbf{v} = \cos\lambda_2(z\,\partial_x - \partial_y) - \lambda_3\partial_z = \lambda_2X_2 - \lambda_3X_3$$

The equations (3.1) and conditions(3.2) take the form respectively

$$\begin{cases} \dot{x} = \cos z\,\lambda_2, \\ \dot{y} = -\lambda_2, \\ \dot{z} = -\lambda_3, \end{cases} \tag{3.7}$$

$$(3.8) \qquad \Omega_1(rot\mathbf{v}) = \partial_z p_2 - \partial_y p_3 + \cos z \partial_x p_3 = 0$$

where

$$(3.9) \qquad \begin{cases} rot\mathbf{v} = \dfrac{1}{\sqrt{detG}}(\partial_z p_2 - \partial_y p_3,\, \partial_x p_3,\, -\partial_x p_2) \\ p_k = \dfrac{\partial T}{\partial \dot{x}^k}|_{\dot{\mathbf{x}}=\mathbf{v}} \end{cases}$$

From (3.9) we deduce that

$$\begin{cases} -I_1 I_2 \lambda_3 = I_1 p_3 + (I_2 - I_1)(p_3 \cos x + \dfrac{p_2 \sin x}{\sin z}) \\ I_1 I_2 \sin z \lambda_2 = \dfrac{I_1 p_2}{\sin z} + (I_2 - I_1) \sin x (p_3 \cos x + \dfrac{p_2 \sin x}{\sin z}) \end{cases}$$

By introducing the change

$$\begin{cases} p_2 = \sin z(\mu_2 \sin x - \mu_1 \cos x) \\ p_3 = \mu_2 \cos x + \mu_1 \sin x, \end{cases}$$

we obtain that the system (3.7)and equation (3.8) admit the representation respectively

$$(3.10) \qquad \begin{cases} \dot{x} = \dfrac{\cot z}{I_1 I_2}(I_1 \mu_1 \cos x - I_2 \mu_2 \sin x), \\ \dot{y} = \dfrac{-1}{I_1 I_2 \sin z}(I_1 \mu_1 \cos x - I_2 \mu_2 \sin x) \\ \dot{z} = \dfrac{1}{I_1 I_2}(I_1 \mu_1 \sin x + I_2 \mu_2 \cos x), \end{cases}$$

$$(3.11) \qquad \sin x(\sin z \partial_z \mu_2 + \cos z \partial_x \mu_1) + \cos x(\cos z \partial_x \mu_2 - \sin z \partial_z \mu_1 - \partial_y \mu_2) = 0$$

Clearly,

$$||\mathbf{v}||^2 = \frac{1}{I_1 I_2}(I_1 \mu_1^2 + I_2 \mu_2^2)$$

Now we shall study the particular case when (3.11) holds in view of the relations

$$(3.12) \qquad \begin{cases} \tan z \partial_z \mu_2 = -\partial_x \mu_1 \\ \partial_x \mu_2 = \tan z \partial_z \mu_1 \end{cases}$$

From the compatibility conditions we obtain the following partial differential equation

$$(3.13) \qquad \sin^2 z \cos z \frac{\partial^2 \mu_j}{\partial z \partial z} + \cos^3 z \frac{\partial^2 \mu_j}{\partial x \partial x} + \sin z \frac{\partial \mu_j}{\partial z} = 0, \quad j = 1, 2$$

Corollary 3.1

Let μ_1 and μ_2 are solutions of the system (3.12), then the function $F = \mu_1 + \imath\mu_2$ is holomorphic function on the complex variable $w = \gamma_2 + \imath\gamma_1 = e^{\imath x}\sin z$.

In fact, after the change $u = \ln\sin z$ from (3.12) we deduced the Cauchy-Riemann equations

$$\begin{cases} \partial_u\mu_2 = -\partial_x\mu_1 \\ \partial_x\mu_2 = \partial_u\mu_1 \end{cases}$$

i.e., the function

$$F(u + \imath x) = F(\ln w)$$

is an holomorphic function, hence

$$\begin{cases} \mu_1 = \Re F = \partial_{\gamma_2}S = \partial_{\gamma_1}\Psi, \\ \mu_2 = \Im F = \partial_{\gamma_1}S = -\partial_{\gamma_2}\Psi \end{cases}$$

as a consequence we obtain that the functions

$$\begin{cases} I_1\omega_1 = \Re F(\ln(\gamma_2 + \imath\gamma_1)) \\ I_2\omega_2 = -\Im F(\ln(\gamma_2 + \imath\gamma_1)) \end{cases}$$

are first integral of (3.3)+(3.4).

Corollary 3.2

If $I_1 = I_2$ then Ψ is a first integral of (3.3) and the function U we determine as follow

$$U = |F(\ln(\gamma_2 + \imath\gamma_1))|^2$$

It is easy to show that the solutions of the equations (3.10) in this case are such that

$$\begin{cases} \displaystyle\int \frac{d(\ln(\gamma_1 + \imath\gamma_2))}{F(\ln(\gamma_1 + \imath\gamma_2))} = \tau - \tau_0 \\ \gamma_3 = \sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)} \\ t = t_0 + \displaystyle\int \frac{d\tau}{\sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)}} \end{cases}$$

Clearly, if μ_1, μ_2 satisfies the Cauchy-Riemann condition then they are solutions of the Laplace equation

$$\frac{\partial^2\mu_j}{\partial u\partial u} + \frac{\partial^2\mu_j}{\partial x\partial x} = 0, \quad j = 1, 2.$$

Hence, if

$$\mu_j = X_j(x)Y_j(u), \quad j = 1, 2 \tag{3.14}$$

then X and Y are solution of the second ordinary differential equation respectively

$$X_j^{''}(x) + \nu^2 X_j(x) = 0, \tag{3.15}$$

and

$$Y_j^{''}(u) - \nu^2 Y_j(u) = 0, \quad j = 1,2 \tag{3.16}$$

where ν is a real constant.

Corollary 3.3

Let μ_1 and μ_2 are solutions of the system (3.13), then

$$\frac{\partial^2 \mu_j}{\partial u \partial u} - \frac{u^2+1}{u-u^3}\frac{\partial \mu_j}{\partial u} + \frac{u^2}{(u^2-1)^2}\frac{\partial^2 \mu_j}{\partial x \partial x} = 0, \quad u = \cos z, \quad j = 1,2$$

If we represent μ_j by the formula (3.16) then X is a solution of the differential equation (3.17). and Y is a solution of the Fuchsian equation

$$Y_j''(u) - \frac{u^2+1}{u-u^3}Y_j'(u) - \frac{\nu^2 u^2}{(u^2-1)^2}Y_j(u) = 0, \quad j = 1,2 \tag{3.17}$$

The proof we obtain after the calculations from (3.19), after the change $u = \cos z$.

Analogously we can prove the following assertion

Corollary 3.4

Let μ_1 and μ_2 are solutions of the system (3.13), then its are solutions of the partial differential equations

$$\frac{\partial^2 \mu_j}{\partial u \partial u} + \frac{1}{u}\frac{\partial \mu_j}{\partial u} + \frac{1}{u^2}\frac{\partial^2 \mu_j}{\partial x \partial x} = 0, \quad u = \sin z, \quad j = 1,2$$

Hence, if (3.14) holds then X satisfies (3.15) and Y is a solution of the Euler ordinary differential equation

$$u^2 Y_j''(u) + u Y_j'(u) - \nu^2 Y_j(u) = 0, \quad j = 1,2 \tag{3.18}$$

where ν is real constant.

Proposition 3.2

The functions

$$\begin{cases} I_1\omega_1 = X_2(x)Y_2(u) \\ I_2\omega_2 = -X_1(x)Y_1(u) \end{cases}$$

where X_1, X_2 are solutions of (3.15) and Y_1, Y_2 are solutions of (3.16) or (3.17) or (3.18), are the first integrals of the system (3.3)+(3.4).

Denoting by $\gamma_1 = \sin z \sin x, \quad \gamma_2 = \sin z \cos x, \quad \gamma_3 = \cos z$ from (3.10) and (3.11) we obtain the relations respectively

$$\begin{cases} \dot{\gamma_1} = \dfrac{I_1}{I_1 I_2}\mu_1\gamma_3 \\ \dot{\gamma_2} = \dfrac{I_2}{I_1 I_2}\mu_2\gamma_3 \\ \dot{\gamma_3} = \dfrac{-1}{I_1 I_2}(I_1\mu_1\gamma_1 + I_2\mu_2\gamma_2) \end{cases} \tag{3.19}$$

$$\sin z(\gamma_3(\frac{\partial\mu_1}{\partial\gamma_2}-\frac{\partial\mu_2}{\partial\gamma_1})-\gamma_2\frac{\partial\mu_1}{\partial\gamma_3}+\gamma_1\frac{\partial\mu_2}{\partial\gamma_3})-\cos x\partial_y\mu_2-\sin x\partial_y\mu_1=0$$

We shall study only the case when

$$\mu_j=\mu_j(x,z),\ j=1,2$$

Hence, we obtain the equation (3.5).

By compare (3.19) with (3.3) we deduce that

$$I_1\omega_1=\mu_2,\quad I_2\omega_2=-\mu_1 \tag{3.20}$$

Corollary 3.5

Let μ_1, μ_2 are such that

$$\mu_j=\frac{\partial S(\gamma_1,\gamma_2)}{\partial\gamma_j},\quad j=1,2$$

then the potential function (3.4) and first integrals are

$$\begin{cases} U=\dfrac{1}{2I_1I_2}(I_1(\dfrac{\partial S}{\partial\gamma_1})^2+I_2(\dfrac{\partial S}{\partial\gamma_2})^2)-h\\ I_1\omega_1=\dfrac{\partial S}{\partial\gamma_2},\\ I_2\omega_2=-\dfrac{\partial S}{\partial\gamma_1}, \end{cases}$$

The following particular cases produces the well known integrable cases[Koz2].

Suslov subcase

If

$$S=C_1\gamma_1+C_2\gamma_2,\quad C_j=const,\ j=1,2$$

then

$$\begin{cases} \mu_1=C_2,\quad \mu_2=C_1\\ U=const. \end{cases}$$

hence we obtain the Suslov subcase

The integration of the equations (3.19) produce the following solutions

$$\begin{cases} \omega_1=\dfrac{C_2}{I_1},\quad \omega_2=-\dfrac{C_1}{I_2}\\ \gamma_1=\dfrac{C_2I_2}{\sqrt{I_1^2C_1^2+I_2^2C_2^2}}\sin\beta\sin(\sqrt{\dfrac{I_1^2C_1^2+I_2^2C_2^2}{I_1I_2}}t+\alpha)+\dfrac{I_1C_1\cos\beta}{\sqrt{I_1^2C_1^2+I_2^2C_2^2}}\\ \gamma_2=\dfrac{C_1I_1}{\sqrt{I_1^2C_1^2+I_2^2C_2^2}}\sin\beta\sin(\sqrt{\dfrac{I_1^2C_1^2+I_2^2C_2^2}{I_1I_2}}t+\alpha)-\dfrac{I_1C_1\cos\beta}{\sqrt{I_1^2C_1^2+I_2^2C_2^2}}\\ \gamma_3=\sin\beta\cos((\sqrt{\dfrac{I_1^2C_1^2+I_2^2C_2^2}{I_1I_2}}t+\alpha) \end{cases}$$

where C_1, C_2, α, β, are the arbitrary real constants.

Kharlamova-Zabelina subcase

If

$$S=\frac{2}{3\sqrt{I_1C_1^2+I_2C_2^2}}(\sqrt{\tilde{h}+C_1\gamma_1+C_2\gamma_2})^3+\frac{C}{2C_1I_1}\gamma_1-\frac{C}{2C_2I_2}\gamma_2$$

where $\tilde{h}$, C_1, C_2, C are arbitrary constants, then

$$\begin{cases}\mu_1=\dfrac{C_1}{\sqrt{I_1C_1^2+I_2C_2^2}}\sqrt{\tilde{h}+C_1\gamma_1+C_2\gamma_2}+\dfrac{C}{2C_1I_1}\\ \mu_2=\dfrac{C_2}{\sqrt{I_1C_1^2+I_2C_2^2}}\sqrt{\tilde{h}+C_1\gamma_1+C_2\gamma_2}-\dfrac{C}{2C_2I_2}\\ U=\tilde{h}+C_1\gamma_1+C_2\gamma_2\end{cases}$$

As a consequence we deduce the Kharlamova-Zabelina subcase.

The solutions of the equation (3.19) give the following solutions

$$\begin{cases}I_1\omega_1=\dfrac{C_2}{\sqrt{I_1C_1^2+I_2C_2^2}}\sqrt{\tilde{h}+C_1\gamma_1+C_2\gamma_2}-\dfrac{C}{2C_2I_2},\\ I_2\omega_2=-\dfrac{C_1}{\sqrt{I_1C_1^2+I_2C_2^2}}\sqrt{\tilde{h}+C_1\gamma_1+C_2\gamma_2}-\dfrac{C}{2C_1I_1},\\ \gamma_1=\dfrac{I_1C_1(\tau^2+C_3)+C_2(CA\tau+C_4)}{I_1C_1^2+I_2C_2^2}=\gamma_1(\tau,C_1,C_2,C_3,C_4)\\ \gamma_2=\dfrac{I_2C_2(\tau^2+C_3)-C_1(CA\tau+C_4)}{I_1C_1^2+I_2C_2^2}=\gamma_2(\tau,C_1,C_2,C_3,C_4)\\ \gamma_3=\sqrt{1-\gamma_1^2(\tau,C_1,C_2,C_3,C_4)-\gamma_2^2(\tau,C_1,C_2,C_3,C_4)}\equiv\sqrt{P_4(\tau,C_1,C_2,C_3,C_4)}\\ t=t_0+\dfrac{I_1I_2}{2}\displaystyle\int\frac{d\tau}{\sqrt{P_4(\tau,C_1,C_2,C_3,C_4)}}\end{cases}$$

where $4C_1C_2A=I_1C_1^2+I_2C_2^2$, and P_4 is a polynomial of four degree in τ.

Kozlov subcase

If we suppose that $I_1=I_2$ and

$$\begin{cases}S=-2Cx+\displaystyle\int D(\gamma_1^2+\gamma_2^2)d(\gamma_1^2+\gamma_2^2)\\ D^2(u)=\dfrac{hu^2+\sqrt{1-u}u-C^2}{u^2}\end{cases}$$

where h and C are arbitrary real constant.

Hence,

$$\begin{cases} \mu_1 = -\dfrac{\gamma_2 C}{\gamma_1^2+\gamma_2^2} + \gamma_1 D(\gamma_1^2+\gamma_2^2) \\ \mu_2 = \dfrac{\gamma_1 C}{\gamma_1^2+\gamma_2^2} + \gamma_2 D(\gamma_1^2+\gamma_2^2) \\ U = h + \sqrt{1-\gamma_1^2-\gamma_2^2} = h + \gamma_3 \end{cases}$$

which correspond to the Kozlov subcase.

The equations (3.10) in this case take the form:

$$\begin{cases} \dot{x} = \dfrac{C\cos z}{\sin^2 z} \\ \dot{y} = \dfrac{-C}{\sin^2 z} \\ \dot{z} = \dfrac{(\gamma_1^2+\gamma_2^2)D(\gamma_1^2+\gamma_2^2)}{\sin z} \end{cases} \tag{3.21}$$

which are easy to integrate.

The solutions of the equation of motions are:

$$\begin{cases} \omega_1 = \dfrac{\gamma_1 C}{\gamma_1^2+\gamma_2^2} + \gamma_2 D(\gamma_1^2+\gamma_2^2) \\ \omega_2 = \dfrac{\gamma_2 C}{\gamma_1^2+\gamma_2^2} - \gamma_1 D(\gamma_1^2+\gamma_2^2) \\ x = x_0 + C\displaystyle\int \frac{\gamma_3 d\gamma_3}{(1-\gamma_3^2)^2 D(1-\gamma_3^2)} = x_0 + C\int \frac{\gamma_3 d\gamma_3}{\sqrt{(1-\gamma_3^2)P_4(\gamma_3,h,C)}} \\ y = y_0 - C\displaystyle\int \frac{d\gamma_3}{(1-\gamma_3^2)^2 D(1-\gamma_3^2)} = y_0 - C\int \frac{d\gamma_3}{\sqrt{(1-\gamma_3^2)P_4(\gamma_3,h,C)}} \\ t = t_0 + I_1 I_2 \displaystyle\int \frac{d\gamma_3}{\sqrt{P_4(\gamma_3,h,C)}} \\ P_4(\gamma_3,h,C) \equiv h\gamma_3^4 - 2\gamma_3^3 - 2h\gamma_3^2 + 2\gamma_3 + h - C^2 \end{cases}$$

where x_0, y_0, h, C, t_0 are arbitrary constants

Tisserand subcase

Another interesting solution of the equation (3.5) are

$$\begin{cases} \mu_1 = \sqrt{h_1 + a_1(\gamma_3^2+\gamma_2^2) + b_1\gamma_1^2 + f_1(\gamma_1)} \\ \mu_2 = \sqrt{h_2 + a_2(\gamma_3^2+\gamma_1^2) + b_2\gamma_2^2 + f_2(\gamma_2)} \end{cases}$$

which produce the following potential function U :

$$U = I_1h_1 + I_2h_2 + (I_1b_1+I_2a_2)\gamma_1^2 + (I_1a_1+I_2b_2)\gamma_2^2 + (I_1a_1+I_2a_2)\gamma_3^2 + I_1f_1(\gamma_1) + I_2f_2(\gamma_2)$$

where a_j, b_j, h_j, $j = 1, 2$ are arbitrary real constants and f_j, $j = 1, 2$ are arbitrary functions.

The case when $f_j(\gamma_j) = \alpha_j \gamma_j$, $j = 1, 2$ was studied in [Okuneba], where α_j, $j = 1, 2$ are real constants.

The case when $f_j = 0$, $j = 1, 2$ is well known as Tisserands case [Koz2].

After integration the equation (3.19) in the Tisserand case we obtain the following solutions

$$\begin{cases} I_1\omega_1 = \sqrt{h_2 + a_2(\gamma_3^2 + \gamma_1^2) + b_2\gamma_2^2} \\ I_2\omega_2 = -\sqrt{h_1 + a_1(\gamma_3^2 + \gamma_2^2) + b_1\gamma_1^2} \\ \gamma_1 = \sqrt{\dfrac{h_1 + a_1}{a_1 - b_1}}\sin(\sqrt{a_1 - b_1}I_1\tau + C_1) = \gamma_1(\tau) \\ \gamma_2 = \sqrt{\dfrac{h_2 + a_2}{a_2 - b_2}}\sin(\sqrt{a_2 - b_2}I_2\tau + C_2) = \gamma_2(\tau) \\ \gamma_3 = \sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)} \\ t = t_0 + I_1 I_2 \displaystyle\int \frac{d\tau}{\sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)}} \end{cases}$$

To conclude the construction the Cartesian approach for heavy rigid body in the Suslov case we analyze the case when **v** is a Kummer vector field.

Kummer subcase

In view of (3.9) we obtain that **v** is a Kummer vector field if

$$\begin{cases} \partial_z p_2 - \partial_y p_3 = \nu \cos\theta \lambda_2 \\ \partial_x p_3 = \nu \lambda_2 \\ \partial_x p_2 = -\nu \lambda_3 \end{cases} \tag{3.22}$$

Hence we deduce that if

$$I_1 = I_2, \quad \nu = 0$$

then the functions p_2, p_3 :

$$\begin{cases} p_2 = C = const. \\ p_3 = \lambda_3(z), \end{cases}$$

are the solutions of (3.22). By considering the relations between p_2, p_3 and μ_1, μ_2 :

$$\mu_1 = p_3 \sin x - \frac{p_2}{\sin z}\cos x \equiv \lambda_3(z)\sin x - \frac{C}{\sin^2 z}\gamma_2$$
$$\mu_2 = p_3 \cos x + \frac{p_2}{\sin z}\sin x \equiv \lambda_3(z)\cos x + \frac{C}{\sin^2 z}\gamma_1.$$

By choosing

$$\lambda_3 = \sin z D(\sin^2 z)$$

we obtain the Kozlov case given above.

To conclude we have the following assertion.

Proposition 3.3

The Cartesian approach for the heavy rigid body in the Suslov case produce the first integrals (3.6).

Example 3.

HEAVY RIGID BODY IN THE VESELOV CASE

In this example we study the problem of non-holonomic dynamics formulated by Veselov in [Veselov] which in certain sense is opposite to the Suslov problem. In this problem we consider the rotational motion of a rigid body around a fixed point and subject to the non-holonomic constraints

$$(\gamma, \omega) \equiv \dot{y} + \cos z \dot{x} = 0 \tag{3.23}$$

Suppose the body rotates in an force field with potential $U(\gamma_1, \gamma_2, \gamma_3)$. Applying the method of Lagrange multipliers we write the equations of motion in the form

$$\begin{cases} I\dot{\omega} = [I\omega \times \omega] + [\gamma \times \dfrac{\partial U}{\partial \gamma}] + \lambda\gamma \\ \dot{\gamma} = [\gamma \times \gamma] \end{cases}$$

where I is a matrix such that $I = diag(I_1, I_2, I_3)$.

The Cartesian approach for this system produce the following equations:

$$\begin{cases} \dot{x} = \lambda_2 \\ \dot{y} = -\cos z \lambda_2 \\ \dot{z} = \lambda_3 \end{cases} \tag{3.24}$$

and

$$\frac{\partial p_3}{\partial x} - \frac{\partial p_1}{\partial z} + \cos z (\frac{\partial p_2}{\partial z} - \frac{\partial p_3}{\partial y}) = 0 \tag{3.25}$$

where

$$\begin{cases} p_1 = I_3 \sin^2 z \lambda_2 \\ p_2 = (I_3 - I_1 + (I_1 - I_2)\cos^2 x) \cos z \sin^2 z \lambda_2 + (I_1 - I_2) \cos x \sin x \sin z \lambda_3 \\ p_3 = (I_2 \sin^2 x + I_1 \cos^2 x)\lambda_3 + (I_2 - I_1) \sin x \cos x \sin z \cos z \lambda_2 \end{cases}$$

Hence

$$
\begin{cases}
\omega_1 = \gamma_2 \dfrac{\lambda_3}{\sin z} - \gamma_1\gamma_3\lambda_2 \\
\omega_2 = -\gamma_1 \dfrac{\lambda_3}{\sin z} - \gamma_2\gamma_3\lambda_2 \\
\omega_3 = \sin^2 z\lambda_2
\end{cases} \tag{3.26}
$$

Clearly, in this case

$$
\begin{aligned}
||\mathbf{v}||^2 =&(I_3\sin^2 z + (I_1\sin^2 x + I_2\cos^2 x)\cos^2 z)\sin^2 z\lambda_2^2 + \\
&(I_2\sin^2 x + I_1\cos^2 x)\lambda_3^2 + 2(I_2 - I_1)\cos x\sin x\cos z\sin z\lambda_2\lambda_3 = 2(U+h)
\end{aligned} \tag{3.27}
$$

or, what is the same

$$
\begin{cases}
||\mathbf{v}||^2 = (\Phi + (I_1 - I_2)\cos x\sin x\cos z\Psi)^2 + (\dfrac{\gamma_1^2}{I_1} + \dfrac{\gamma_2^2}{I_2} + \dfrac{\gamma_3^2}{I_3})I_1I_2I_3\Psi^2 = 2(U+h) \\
\Phi = \sqrt{I_3\sin^2 z + (I_1\sin^2 x + I_2\cos^2 x)\cos^2 z}\sin^2 z\lambda_2 \\
\Psi = \sqrt{I_3\sin^2 z + (I_1\sin^2 x + I_2\cos^2 x)\cos^2 z}\lambda_3
\end{cases}
$$

In particular if

$$
\begin{aligned}
&p_j = p_j(x,z), \quad j = 1,2,3. \\
&I_1 = I_2
\end{aligned}
$$

from (3.25)+(3.26) we obtain the equation

$$
\frac{I_3}{I_2}\frac{\partial\lambda_3}{\partial x} - \sqrt{I_3\sin^2 z + I_1\cos^2 z}\frac{\partial\Phi}{\partial z} = 0,
$$

for which the functions

$$
\Phi = C = const., \quad \lambda_3 = \lambda_3(z) \tag{3.28}
$$

are its solutions.

Hence we easily deduced the proof of the following result

Proposition 3.4

Let us suppose that

$$
U = U(\gamma_3)
$$

then the system (3.25)+(3.28) admits the following solutions

$$
\begin{cases}
x = x_0 - \dfrac{C}{I_3}\displaystyle\int \dfrac{d\gamma_3}{(1-\gamma_3^2)\sqrt{I_3 + (I_1 - I_3)\gamma_3^2}\sqrt{2(U(\gamma_3)+h)(1-\gamma_3^2) - C^2}} \\
y = y_0 - \dfrac{C}{I_3}\displaystyle\int \dfrac{\gamma_3 d\gamma_3}{(1-\gamma_3^2)\sqrt{I_3 + (I_1 - I_3)\gamma_3^2}\sqrt{2(U(\gamma_3)+h)(1-\gamma_3^2) - C^2}} \\
t = t_0 - \displaystyle\int \dfrac{d\gamma_3}{\sqrt{2(U(\gamma_3)+h)(1-\gamma_3^2) - C^2}}
\end{cases}
$$

Clearly, in this case there exist the first integral

$$(I_3 \sin^2 z + I_1 \cos^2 z)\omega_3^2 = C^2.$$

The proof follow from (3.26)+(3.28).

It is interesting to observe that from (3.25) after the change

$$\begin{cases} \lambda_2 \cos z \sin z = \dfrac{\cos z \cos x\, \xi}{I_3 + ((I_2 - I_3) \cos^2 z} - \dfrac{\cos z \sin x\, \eta}{I_3 + ((I_1 - I_3) \cos^2 z} \\ \lambda_3 = \dfrac{\cos z \cos x\, \eta}{I_3 + ((I_1 - I_3) \cos^2 z} + \dfrac{\cos z \sin\, x\, \xi}{I_3 + ((I_2 - I_3) \cos^2 z} \end{cases}$$

and by require that

$$p_j = p_j(x, z), \quad j = 1, 2$$

we deduce the following equation

$$\sin x(\frac{\partial_x \xi}{1 + (\alpha - 1)\sin^2 z} + \tan z \partial_z \eta) + \cos x(\frac{\partial_x \eta}{1 + (\beta - 1)\sin^2 z} - \tan z \partial_z \xi) = 0,$$

where $\alpha = \frac{I_3}{I_2}, \quad \beta = \frac{I_3}{I_1}$. If $\alpha = \beta = 1$ then this equation coincide with (3.11).

Finally it is interesting to observe that the construction the Cartesian approach for the Federov case [Federov],i.e.,

$$(\omega, \gamma) = a$$

it is necessary in the above example make the change $y = Y + at$, $a = const.$.

4. CARTESIAN APPROACH FOR NON-HOLONOMIC SYSTEM WITH FIVE DEGREE OF FREEDOM AND TWO CONSTRAINTS

This case we shall illustrate in one of the interesting non-holonomic mechanical system: the rattleback.

Example 4.

THE RATTLEBACK

The rattleback's amazing mechanical behaviour is a convex asymmetric rigid body rolling without sliding on a horizontal plane. It is known for its ability to spin in one direction and to resist spinning in the opposite direction for some parameters values, and for others values to exhibit multiple reversals. Basic references on the rattleback are [Wal, Mar, Kar, Bor,Tsyg].

Introduce the Euler angles ψ, ϕ, θ using the principal axis body frame relative to an inertial reference frame. These angles together with two horizontal coordinates x, y of the center of mass are coordinates in the configuration space $\mathcal{Q} = SO(3) \times \mathrm{R}^2$ of the rattleback.

The Lagrangian of the rattleback is computed to be

$$
\begin{aligned}
L =& \frac{1}{2}(I_1 \cos^2 \psi + I_2 \sin^2 \psi + m(\Gamma_1 \cos\theta - \zeta \sin\theta)^2)\dot{\theta}^2 \\
& \frac{1}{2}(I_1 \sin^2 \psi + I_2 \cos^2 \psi) \sin^2 \theta) + I_3 \cos^2 \theta)\dot{\phi}^2 \\
& + \frac{1}{2}(I_3 + m\Gamma_2^2 \sin^2 \theta)\dot{\psi}^2 + \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \\
& + m(\Gamma_1 \cos\theta - \zeta \sin\theta)\Gamma_2 \sin\theta\dot{\theta}\dot{\psi} + (I_1 - I_2) \sin\theta \sin\psi \cos\psi\dot{\theta}\dot{\phi} \\
& C \cos\theta\dot{\phi}\dot{\psi} + mg(\Gamma_1 \sin\theta + \zeta \cos\theta)
\end{aligned}
$$

where I_1, I_2, I_3 are the principal moments of inertia of the body, m is the total mass of the body,

$$\Gamma_1 = \xi \sin\psi + \eta \cos\psi, \quad \Gamma_2 = \xi \cos\psi - \eta \sin\psi$$

$(\xi(\theta, \psi), \eta(\theta, \psi), \zeta(\theta, \psi))$ are the coordinates of the point of contact relative to the body frame.

The shape of the body is encoded by the functions ξ, η and ζ. The constraints are

$$
\left\{
\begin{array}{l}
\dot{x} - \alpha_1\dot{\theta} - \alpha_2\dot{\psi} - \alpha_3\dot{\phi} = 0 \\
\dot{y} - \beta_1\dot{\theta} - \beta_2\dot{\psi} - \beta_3\dot{\phi} = 0
\end{array}
\right. \tag{4.1}
$$

where

$$
\left\{
\begin{array}{l}
\alpha_1 = -(\Gamma_1 \sin\theta + \zeta \cos\theta) \sin\phi, \\
\alpha_2 = \Gamma_2 \cos\theta \sin\phi + \Gamma_1 \cos\phi, \\
\alpha_3 = \Gamma_2 \sin\phi + (\Gamma_1 \cos\theta - \zeta \sin\theta) \cos\phi, \\
\beta_k = -\dfrac{\partial\alpha_k}{\partial\phi}, \quad k = 1, 2, 3
\end{array}
\right.
$$

To determine the Cartesian approach of the Rattleback we first determine the vector field $\mathbf{v}$.

The 1-forms Ω_j, $j = 1, ..., 5$ in this case are the following

$$
\left\{
\begin{array}{l}
\Omega_1 = dx - \alpha_1 d\theta - \alpha_2 d\psi - \alpha_3 d\phi, \\
\Omega_2 = dy - \beta_1 d\theta - \beta_2 d\psi - \beta_3 d\phi, \\
\Omega_3 = d\theta, \quad \Omega_4 = d\psi, \quad \Omega_5 = d\phi
\end{array}
\right.
$$

Hence $\Upsilon = 1$ and

$$
\left\{
\begin{array}{l}
\mathbf{v} = \lambda_3 X_3 + \lambda_4 X_4 + \lambda_5 X_5 \\
X_3 = \alpha_1\partial_x + \beta_1\partial_y + \partial_\theta \\
X_4 = \alpha_2\partial_x + \beta_2\partial_y + \partial_\psi \\
X_5 = \alpha_3\partial_x + \beta_3\partial_y + \partial_\phi
\end{array}
\right. \tag{4.2.}
$$

The rattleback in the Kummer case

We now proceed to the consideration of the particular case for which ξ, η and ζ are constants. It is easy to show that under this consideration the vector field X_1, X_2, X_3 generated a three dimensional Abelian Lie algebra.

It is evident that the rattleback equations of motion in this particular case formally contain the equations of the heavy rigid body in the singular case

$$m \to 0, \quad mg \to l, \quad l \neq 0$$

Let (x^1, x^2, x^3, x^4, x^5) be a new set of variables derived from x,, y, θ, ψ, ϕ by the transformation

$$\begin{cases} \psi = x^1 \\ \phi = x^2, \\ \theta = x^3 \\ y + \zeta \sin\theta\cos\phi - \Gamma_1 \cos\theta\sin\phi - \Gamma_2\cos\phi = x^4 \\ x + \zeta\sin\theta\sin\phi + \Gamma_1\cos\theta\cos\phi - \Gamma_2\sin\phi = x^5 \end{cases}$$

The vector field **v** and the constraints on account of this change, take respectively the form

$$\mathbf{v} = a(x^1, .., x^5))\partial_{x^1} + b(x^1, ..., x^5)\partial_{x^2} + c(x^1, ..., x^5)\partial_{x^3}$$

$$(4.3) \qquad \begin{cases} \dot{x}^4 = 0, \\ \dot{x}^5 = 0 \end{cases},$$

By the above transformation the Lagrangian function L is changed into a new Lagrangian

$$\check{L} = \frac{1}{2}\sum_{j,k=1}^{5} G_{jk}(x^1, ..., x^5)\dot{x}^j\dot{x}^k + mg(\Gamma_1\sin\theta + \zeta\cos\theta),$$

where $G = (G_{jk})$ is the Riemann metric which is easy to calculate.

We shall now determine the Cartesian approach under the given conditions.

Proposition 4.1

The vector field **v** given by the formula (4.2) is a Kummer vector field.

Proof. In fact, by considering that in this case the 1-form associated to the vector field **v** is the following

$$\begin{cases} \sigma = p_1 dx^1 + p_2 dx^2 + p_3 dx^3 \\ p_k = \sum_{j=1}^{5} G_{jk}(x)\mathbf{v}(x^j), \quad k = 1, 2, .., 5 \end{cases}$$

then

$$\imath_{\mathbf{v}} d\sigma = \sum_{j=1}^{5} \Lambda_j dx^j$$

$$\begin{cases} \Lambda_1 = (\frac{\partial p_1}{\partial x^2} - \frac{\partial p_2}{\partial x^1})b + (\frac{\partial p_1}{\partial x^3} - \frac{\partial p_3}{\partial x^1})c \\ \Lambda_2 = (\frac{\partial p_2}{\partial x^3} - \frac{\partial p_3}{\partial x^2})c + (\frac{\partial p_2}{\partial x^1} - \frac{\partial p_1}{\partial x^2})a \\ \Lambda_3 = (\frac{\partial p_3}{\partial x^2} - \frac{\partial p_2}{\partial x^3})b + (\frac{\partial p_3}{\partial x^1} - \frac{\partial p_1}{\partial x^3})a \\ \Lambda_4 = -\frac{\partial p_1}{\partial x^4}a - \frac{\partial p_2}{\partial x^4}b - \frac{\partial p_3}{\partial x^4}c \\ \Lambda_5 = -\frac{\partial p_1}{\partial x^5}a - \frac{\partial p_2}{\partial x^5}b - \frac{\partial p_3}{\partial x^5}c \end{cases}$$

Let $\mathbf{v}(x)$ and $rot\mathbf{v}(x)$ are the following vectors

$$\mathbf{v}(x) = (a,\, b,\, c)$$

$$rot\mathbf{v} = \frac{1}{\sqrt{detG}}(\frac{\partial p_3}{\partial x^2} - \frac{\partial p_2}{\partial x^3}, \frac{\partial p_1}{\partial x^3} - \frac{\partial p_3}{\partial x^1}, \frac{\partial p_2}{\partial x^1} - \frac{\partial p_1}{\partial x^2})$$

We have therefore that the equations (2.2)+(2.3) take the form respectively

$$\begin{cases} \dot{x}^1 = a(x^1, x^2, x^3, C_4, C_5) \\ \dot{x}^2 = b(x^1, x^2, x^3, C_4, C_5) \\ \dot{x}^3 = c(x^1, x^2, x^3, C_4, C_5) \end{cases} \tag{4.4}$$

$$[\mathbf{v} \times rot\mathbf{v}(x)] = 0, \tag{4.5}$$

hence the constructed vector field is a Kummer vector field.

The rattleback in the Suslov case

We shall now consider the motion of the rattleback with the following set of complementary conditions

$$\begin{cases} \xi = const., \quad \eta = const., \quad \zeta = const. \\ (\Gamma_1^2 + \Gamma_2^2)\zeta = 0 \\ \dot{\psi} + \cos\theta\dot{\phi} = 0 \end{cases}$$

This subcase we call *the Rattleback in the Suslov case.*

Similarly to the above case we have that the vector field $\mathbf{v}$ and equations (3.27) takes the form respectively

$$\mathbf{v} = -\cos x^3 b(x^1, .., x^5))\partial_{x^1} + b(x^1, ..., x^5)\partial_{x^2} + c(x^1, ..., x^5)\partial_{x^3}$$

$$\begin{cases} \dot{x}^4 = 0, \\ \dot{x}^5 = 0 \\ \dot{x}^1 + \cos x^3 \dot{x}^2 = 0 \end{cases}, \tag{4.6}$$

$$(4.7)\qquad \begin{cases} \dfrac{\partial p_2}{\partial x^1} b + \dfrac{\partial p_3}{\partial x^1} c = 0 \\ (\dfrac{\partial p_2}{\partial x^3} - \dfrac{\partial p_3}{\partial x^2}) c + (\dfrac{\partial p_2}{\partial x^1}) a = 0 \\ (\dfrac{\partial p_3}{\partial x^2} - \dfrac{\partial p_2}{\partial x^3}) b + \dfrac{\partial p_3}{\partial x^1} a = 0 \end{cases}$$

Now the function p_1, p_2, p_3 are such that

$$\begin{cases} \Gamma_1^2 + \Gamma_2^2 = 0 \\ p_1 = 0 \\ p_2 = (I_1 \sin^2 \psi + I_2 \cos^2 \psi + m\zeta^2) \sin^2 \theta b \\ p_3 = (I_1 - I_2) \sin\theta \sin\psi b + (I_1 \sin^2 \psi + I_2 \cos^2 \psi + m\zeta^2) c \end{cases}$$

and

$$(4.8)\qquad \begin{cases} \zeta = 0 \\ p_1 = 0 \\ p_2 = (I_1 \sin^2 \psi + I_2 \cos^2 \psi + m\Gamma_2^2) \sin^2 \theta b \\ p_3 = ((I_1 - I_2) \sin\theta \sin\psi - m\Gamma_1\Gamma_2 \sin\theta) b + (I_1 \sin^2 \psi + I_2 \cos^2 \psi + m\Gamma_1^2) c \end{cases}$$

From these equations it is evident that in the case when

$$\begin{cases} \Gamma_1^2 + \Gamma_2^2 = 0 \\ I_1 = I_2 \end{cases}$$

we have that

$$p_2 = (I_1 + m\zeta^2) \sin^2 x^3 b, \; p_3 = (I_1 + m\zeta^2) c$$

hence (4.5) are therefore satisfied by this functions if

$$\begin{cases} \sin^2 x^3 b^2 + c^2 = K(x^2, x^3) \\ \dfrac{\partial c}{\partial x^2} - \cos x^3 \dfrac{\partial c}{\partial x^1} = \dfrac{\partial \sin^2 x^3 b,}{\partial x^3} \end{cases}$$

where K is an arbitrary function on the variables x^2, x^3.

Hence on taking

$$\sin^2 x^3 b^2 = A(x^2), \quad c = \sqrt{K(x^2, x^3) - A(x^2)} = c(x^3)$$

we see that (4.5) hold. The equations generated by $\mathbf{v}$ in this case are

$$\begin{cases} \dot{x}^1 = -\dfrac{A \cos x^3}{\sin^2 x^3} \\ \dot{x}^2 = -\dfrac{A}{\sin^2 x^3} \\ \dot{x}^3 = c(x^3) \end{cases}$$

which contain as a particular case the system (3.21).

The Rattleback in the general case

For the general case, i.e., when the ξ, η and ζ are functions on the variables θ and ψ the Cartesian approach produce the following equations

$$\dot{x}^k = \mathbf{v}(x^k), \quad k = 1, 2, .., 5$$

$$\begin{cases} \sum_{j=1}^{5} \Big(\frac{\partial p_1}{\partial x^j} - \frac{\partial p_j}{\partial x^1} + \alpha_2(\frac{\partial p_4}{\partial x^j} - \frac{\partial p_j}{\partial x^4}) + \beta_2(\frac{\partial p_5}{\partial x^j} - \frac{\partial p_j}{\partial x^5})\Big)\mathbf{v}(x^j) = 0 \\ \sum_{j=1}^{5} \Big(\frac{\partial p_2}{\partial x^j} - \frac{\partial p_j}{\partial x^2} + \alpha_3(\frac{\partial p_4}{\partial x^j} - \frac{\partial p_j}{\partial x^4}) + \beta_3(\frac{\partial p_5}{\partial x^j} - \frac{\partial p_j}{\partial x^5})\Big)\mathbf{v}(x^j) = 0 \\ \sum_{j=1}^{5} \Big(\frac{\partial p_3}{\partial x^j} - \frac{\partial p_j}{\partial x^1} + \alpha_1(\frac{\partial p_4}{\partial x^j} - \frac{\partial p_j}{\partial x^4}) + \beta_1(\frac{\partial p_5}{\partial x^j} - \frac{\partial p_j}{\partial x^5})\Big)\mathbf{v}(x^j) = 0 \end{cases}$$

where

$$\begin{cases} \psi = x^1, \quad \phi = x^2, \\ \theta = x^3, \quad y = x^4, \quad x = x^5 \end{cases}$$

and $\mathbf{v}$ is given by the formula (4.2).

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